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# On Weak and Strong Convergence to Equilibrium for Solutions to the Linear Boltzmann Equation

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This paper considers the linear space-inhomogeneous Boltzmann equation for a distribution function in a bounded domain with general boundary conditions together with an external potential force. The paper gives results on strong convergence to equilibrium, when $t \to \infty$, for general initial data; first in the cutoff case, and then for infinite-range collision forces. The proofs are based on the properties of translation continuity and weak convergence to equilibrium. To handle these problems general $H$-theorems (concerning monotonicity in time of convex entropy functionals) are presented. Furthermore, the paper gives general results on collision invariants, i.e., on functions satisfying detailed balance relations in a binary collision.

**KEY WORDS**: Linear Boltzmann equation; transport equation; initial boundary value problem; external force; boundary conditions; entropy function; $H$-functional; detailed balance; collision invariants; convergence to equilibrium; infinite-range collisions.

## INTRODUCTION

The linear Boltzmann equation is frequently used for mathematical modeling in physics. One fundamental question concerns the large-time behavior of the function representing the distribution of particles, in particular the problem of convergence toward an equilibrium solution, which will be studied in this paper.

We shall consider the space-inhomogeneous transport equation for a distribution function $f(\mathbf{x}, \mathbf{v}, t)$ (describing, for instance, a neutron distribution) depending on a space variable $\mathbf{x} = (x_1, x_2, x_3)$ in a nonmultiplying, nonabsorbing (i.e., purely scattering) body $D$, and depending on a velocity

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variable $\mathbf{v}=(v_1, v_2, v_3) \in V=\mathbb{R}^3$ and time variable $t \in \mathbb{R}_+$. Here we assume $D=\bar{D}$ to be a closed, bounded domain in $\mathbb{R}^3$ with (piecewise) $C^1$-boundary $\Gamma=\partial D$. In the case of an external force $\mathbf{a}=\mathbf{a}(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3$, the transport equation in the strong form is

$$\frac{\partial f}{\partial t}(\mathbf{x}, \mathbf{v}, t)+\mathbf{v} \cdot \operatorname{grad}_{\mathbf{x}} f(\mathbf{x}, \mathbf{v}, t)+\mathbf{a} \cdot \operatorname{grad}_{\mathbf{v}} f(\mathbf{x}, \mathbf{v}, t)=(Qf)(\mathbf{x}, \mathbf{v}, t)$$
$$\mathbf{x} \in D \backslash \Gamma, \qquad \mathbf{v} \in V, \qquad t \in \mathbb{R}_+ \tag{1}$$

supplemented with initial data

$$\lim_{t \downarrow 0} f(\mathbf{x}, \mathbf{v}, t)=F_0(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in D, \quad \mathbf{v} \in V \tag{2}$$

and some boundary conditions, which in a general case can be written (ref. 8, p. 107)

$$|\mathbf{n} \cdot \mathbf{v}|\, f(\mathbf{x}, \mathbf{v}, t)=\int_{\mathbf{n} \cdot \mathbf{v}'>0} R(\mathbf{x}, \mathbf{v}' \rightarrow \mathbf{v})\, f(\mathbf{x}, \mathbf{v}', t)|\mathbf{n} \cdot \mathbf{v}'|\, d\mathbf{v}',$$
$$\mathbf{x} \in \Gamma, \quad \mathbf{n} \cdot \mathbf{v}<0, \quad t \geqslant 0 \tag{3}$$

Here $\mathbf{n}=\mathbf{n}(\mathbf{x})$ is the unit outward normal vector at $\mathbf{x} \in \Gamma=\partial D$ and $R$ is a given nonnegative function. For instance, in the case of specular reflection $R(\mathbf{x}, \mathbf{v}' \rightarrow \mathbf{v})=\delta(\mathbf{v}-\mathbf{v}'+2\mathbf{n}(\mathbf{n} \cdot \mathbf{v}'))$, where $\delta$ is the usual Dirac measure, and in the case of diffuse reflection $R(\mathbf{x}, \mathbf{v}' \rightarrow \mathbf{v})=|\mathbf{n} \cdot \mathbf{v}|\, M(\mathbf{x}, \mathbf{v})$, where $M(\mathbf{x}, \mathbf{v})$ is a local Maxwell distribution function.

For a nonabsorbing boundary the function $R$ in (3) is supposed to satisfy [8]

$$\int_{\mathbf{n} \cdot \mathbf{v}<0} R(\mathbf{x}, \mathbf{v}' \rightarrow \mathbf{v})\, d\mathbf{v}=1, \qquad \mathbf{x} \in \Gamma, \quad \mathbf{n} \cdot \mathbf{v}'>0 \tag{4}$$

The collision term in (1) can be written[8]

$$(Qf)(\mathbf{x}, \mathbf{v}, t)$$
$$=\int_V \int_\Omega [\psi(\mathbf{x}, \mathbf{v}'_*)\, f(\mathbf{x}, \mathbf{v}', t)-\psi(\mathbf{x}, \mathbf{v}_*)\, f(\mathbf{x}, \mathbf{v}, t)] \cdot B(\theta, w)\, d\theta\, d\zeta\, d\mathbf{v}_* \tag{5}$$

where $\psi \geqslant 0$ is a known distribution function. Here, $\mathbf{v}$ and $\mathbf{v}_*$ are the velocities before, and $\mathbf{v}'$ and $\mathbf{v}'_*$ are the velocities after a binary collision. $\Omega$ is the impact plane $\{(r, \zeta)\colon\ 0 \leqslant r \leqslant \hat{R},\ 0 \leqslant \zeta<2\pi)\}$, which also can be parametrized by the usual solid-angle representation $\{(\theta, \zeta)\colon\ 0 \leqslant \theta<\hat{\theta},$

$0 \leqslant \zeta < 2\pi\}$. In the cutoff case, $\Omega$ is bounded, that is, $\hat{R} < \infty$, or $\hat{\theta} < \pi/2$; but in the case of infinite-range forces, $\Omega$ is the whole plane, i.e., $\hat{\theta} = \pi/2$. The function $B$ is given by $B(\theta, w) = wr|\partial r/\partial\theta|$, where $r = r(\theta, w)$ is computed through the relevant law of interaction, and $w = |\mathbf{v} - \mathbf{v}_*|$. (For details, see refs. 8 and 28; see also refs. 6 and 15).

In many cases of physical interest the function $B(\theta, w)$ has a nonintegrable singularity for $\theta = \pi/2$; for instance, with inverse $k$th power forces, where

$$B(\theta, w) = w^{\gamma} b(\theta) \tag{6}$$

with $\gamma = (k-5)/(k-1)$, $3 < k < \infty$, and[8,28]

$$b(\theta) = \mathcal{O}((\pi/2 - \theta)^{-(k+1)/(k-1)}), \qquad \theta \to \pi/2^-,$$

For that reason most authors have modified the function $B$, for instance by cutoffs of Grad type, thus only allowing forces of essentially finite range in the collision term. (For a discussion of such works, see refs. 21.)

In connection with transforming problem (1)–(3) into a purely integral form we shall, in the case of an external force $\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{v})$, be concerned with the solution $\mathbf{x} = \mathbf{x}(t) \equiv \mathbf{x}(\mathbf{y}, \mathbf{u}, t)$, $\mathbf{v} = \mathbf{v}(t) \equiv \mathbf{v}(\mathbf{y}, \mathbf{u}, t)$ to the *characteristic problem* of the streaming operator:

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{v}(t), \qquad && \mathbf{x}(0) = \mathbf{y} \\ \frac{d\mathbf{v}}{dt} &= \mathbf{a}(\mathbf{x}, \mathbf{v}), \qquad && \mathbf{v}(0) = \mathbf{u} \end{aligned} \tag{7}$$

In the rest of the paper we assume the following hypothesis.

**Hypothesis CP.** 1. There exists a unique, locally absolutely continuous function on $\mathbb{R}$ satisfying (7) for a.e. $t \in \mathbb{R}$.

(2) The Jacobian of the transformation $(\mathbf{y}, \mathbf{u}) \mapsto (\mathbf{x}(t), \mathbf{v}(t))$, $(\mathbf{y}, \mathbf{u}) \in \bar{D} \times V$, is equal to unity for every $t \in \mathbb{R}_+$.

By this assumption we can (formally) reformulate Eq. (1) (using differentiation along the characteristics)

$$\frac{d}{dt}(f(\mathbf{x}(t), \mathbf{v}(t), t)) = (Qf)(\mathbf{x}(t), \mathbf{v}(t), t) \tag{8}$$

*Remark.* The assumption in Hypothesis CP2 is found to be equivalent to the assumption that $\mathbf{a}(\mathbf{x}, \mathbf{v})$ is divergence-free in $\mathbf{v}$; see ref. 24 for further references.

The purpose of this paper is to give results on strong convergence to equilibrium when $t \to \infty$, first in the cutoff case (Section 4) and then for infinite-range forces (Section 5). The proofs are based on a translation continuity property, together with results on weak convergence to equilibrium. These results are derived in Section 4 for the cutoff case, and then transformed to the infinite-range case in Section 5. To handle those problems we present $H$-theorems with general convex functionals of our solutions, first for the cutoff case in Section 3, and then for the noncutoff case in Section 5. We will also in Section 2 present general results on collision invariants for functions satisfying detailed balance relations in binary collisions. We collect in Section 1 some of our earlier results on the existence of solutions to the linear Boltzmann equation with general boundary conditions.

## 1. PRELIMINARIES

In the case of cutoff in the impact parameters, i.e., $\hat{R} < \infty$ or $\hat{\theta} < \pi/2$, the collision term (5) in Eq. (1) can be separated into two terms, "a gain term" and "a loss term." A common way to write the collision term is the following (see ref. 8, and also refs. 21–24):

$$(Qf)(\mathbf{x}, \mathbf{v}, t) = \int_V K(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) f(\mathbf{x}, \mathbf{v}', t)\, d\mathbf{v}' - L(\mathbf{x}, \mathbf{v}) f(\mathbf{x}, \mathbf{v}, t) \quad (1.1)$$

where

$$L(\mathbf{x}, \mathbf{v}) = \int_V K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, d\mathbf{v}' \quad (1.2)$$

The collision frequency $L$ is coupled to the functions $\psi$ and $B$ in (5) by the relation

$$L(\mathbf{x}, \mathbf{v}) = \int_V \int_\Omega \psi(\mathbf{x}, \mathbf{v}_*)\, B(\theta, w)\, d\theta\, d\zeta\, d\mathbf{v}_* \quad (1.3)$$

where $w = |\mathbf{v} - \mathbf{v}_*|$. We assume (for simplicity) that the collision kernel $K$ vanishes on $\Gamma$ and outside $D$.

Let $\Gamma_+ = \Gamma_+(\mathbf{v}) = \{\mathbf{x} \in \Gamma; \mathbf{n} \cdot \mathbf{v} > 0\}$, $\Gamma_- = \Gamma_-(\mathbf{v}) = \{\mathbf{x} \in \Gamma; \mathbf{n} \cdot \mathbf{v} < 0\}$, where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit, outward normal. Let for given $\mathbf{y} \in D \backslash \Gamma_-$, $\mathbf{u} \in V$,

$$t_b = t_b(\mathbf{y}, \mathbf{u}) = \inf\{s > 0; \mathbf{x}(-s) \in \Gamma = \partial D, \mathbf{x}(0) = \mathbf{y}, \mathbf{v}(0) = \mathbf{u}\} \quad (1.4)$$

representing the time for a particle going from the boundary to the point $\mathbf{y}$ following the characteristic curve (see Hypothesis CP).

In this section the linear Boltzmann equation (1) with (2)–(4) and (1.1) is studied in two integrated forms, the mild form [Eq. (1.5) below], and the exponential form (1.6), which both formally can be derived from the equations above. Using, for $t \in \mathbb{R}_+$, a.e. $(\mathbf{y}, \mathbf{u}) = (\mathbf{x}(0), \mathbf{v}(0)) \in D \times V$, the notation

$$\tilde{f}(\mathbf{y}, \mathbf{u}, t) = \begin{cases} F_0(\mathbf{x}(-t), \mathbf{v}(-t)), & 0 \leqslant t \leqslant t_b \\ f(\mathbf{x}(-t_b), \mathbf{v}(-t_b), t - t_b), & t > t_b \end{cases}$$

where $\mathbf{x}(-t_b) \equiv \mathbf{x}(\mathbf{y}, \mathbf{u}, -t_b) \in \Gamma_-(\mathbf{v})$, $\mathbf{v} = \mathbf{v}(\mathbf{y}, \mathbf{u}, -t_b)$, the *mild form* of the equation is

$$f(\mathbf{x}(t), \mathbf{v}(t), t) = \tilde{f}(\mathbf{x}(t), \mathbf{v}(t), t) + \int_0^t (Qf)(\mathbf{x}(\tau), \mathbf{v}(\tau), \tau)\, d\tau \qquad (1.5)$$

and the *exponential form is*

$$\begin{aligned} f(\mathbf{x}(t), \mathbf{v}(t), t) = {} & \tilde{f}(\mathbf{x}(t), \mathbf{v}(t), t) \exp\left[ -\int_0^t L(\mathbf{x}(s), \mathbf{v}(s))\, ds \right] \\ & + \int_0^t \exp\left( -\int_\tau^t L(\mathbf{x}(s), \mathbf{v}(s))\, ds \right) \\ & \times \int_V K(\mathbf{x}(\tau), \mathbf{v}' \to \mathbf{v}(\tau))\, f(\mathbf{x}(\tau), \mathbf{v}', \tau)\, d\mathbf{v}'\, d\tau \end{aligned} \qquad (1.6)$$

*Remark*. One finds that $f$ is a mild solution if and only if the exponential form holds; see ref. 21, and see also ref. 25.

To *construct solutions* to the linear Boltzmann equation with external forces together with general boundary conditions, iterate functions $f_n = f_n(\mathbf{x}, \mathbf{v}, t)$, $n = 0, 1, 2,...$, are defined recursively as follows:

(a) $f_0(\mathbf{x}, \mathbf{v}, t) \equiv 0, \quad \mathbf{x} \in \mathbb{R}^3, \quad \mathbf{v} \in V, \quad t \in \mathbb{R}_+$

(b) $$f_{n+1}(\mathbf{x}_b, \mathbf{v}, t) = \int_{\mathbf{n} \cdot \mathbf{v}' > 0} \frac{|\mathbf{n}\mathbf{v}'|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x}_b, \mathbf{v}' \to \mathbf{v})\, f_n(\mathbf{x}_b, \mathbf{v}', t)\, d\mathbf{v}'$$

$$\mathbf{x}_b \in \Gamma_-(\mathbf{v}), \qquad \mathbf{n} \cdot \mathbf{v} < 0, \qquad t \in \mathbb{R}_+ \qquad (1.7)$$

(c) $f_{n+1}(\mathbf{y}, \mathbf{u}, t)$

$$\begin{aligned} &= \tilde{f}_{n+1}(\mathbf{y}, \mathbf{u}, t) \exp\left[ -\int_0^t L(\mathbf{x}(s-t), \mathbf{v}(s-t))\, ds \right] \\ &\quad + \int_0^t \exp\left[ -\int_\tau^t L(\mathbf{x}(s-t), \mathbf{v}(s-t))\, ds \right] \int_V K(\mathbf{x}(-t), \mathbf{v}' \to \mathbf{v}(\tau - t)) \\ &\quad \times f_n(\mathbf{x}(\tau - t), \mathbf{v}', \tau)\, d\mathbf{v}'\, d\tau \quad \text{a.e. } (\mathbf{y}, \mathbf{u}) \in (D \backslash \Gamma_-) \times V,\ t > 0 \end{aligned}$$

where

$$\tilde{f}_{n+1}(\mathbf{y}, \mathbf{u}, t) = \begin{cases} F_0(\mathbf{x}(-t), \mathbf{v}(-t)), & 0 \leqslant t \leqslant t_b \\ f_{n+1}(\mathbf{x}(-t_b), \mathbf{v}(-t_b), t - t_b), & t > t_b \end{cases}$$

with $\mathbf{x}(0) = \mathbf{y}$, $\mathbf{v}(0) = \mathbf{u}$ and $\mathbf{x}_b = \mathbf{x}(-t_b) \in \Gamma_-(\mathbf{v})$, $\mathbf{v} = \mathbf{v}(-t_b)$. Let also, for simplicity,

$$f_n(\mathbf{x}, \mathbf{v}, t) \equiv 0, \qquad \mathbf{x} \in \mathbb{R}^3 \backslash D, \quad \mathbf{v} \in V, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}$$

Now we first formulate a monotonicity result for the iterates.[23]

**Lemma.** If $F_0$, $K$, and $R$ are nonnegative functions, then the iterates $f_n$ defined by (1.7) satisfy

$$f_{n+1}(\mathbf{x}, \mathbf{v}, t) \geqslant f_n(\mathbf{x}, \mathbf{v}, t), \qquad n \in \mathbb{N}, \quad \mathbf{x} \in \mathbb{R}^3, \quad \mathbf{v} \in V, \quad t \in \mathbb{R}_+ \tag{1.8}$$

Then we can formulate an *existence* theorem about mild solutions to the initial-boundary problem. As usual, $L^1_+(D \times V)$ denotes the almost everywhere nonnegative functions in $L^1(D \times V)$.

**Theorem.** Assume that $R(\mathbf{x}, \mathbf{v}' \to \mathbf{v})$, $L(\mathbf{x}, \mathbf{v})$, and $K(\mathbf{x}, \mathbf{v}' \to \mathbf{v})$ are nonnegative, measurable functions, such that (4) and (1.2) hold, and $L(\mathbf{x}, \mathbf{v}) \in L^1_{\text{loc}}(D \times V)$. If $F_0 \in L^1_+(D \times V)$, then there exists a global mild solution $\mathbf{f}(\mathbf{x}, \mathbf{v}, t)$ (i.e., defined for $t > 0$) to the problem (1)–(3) with (1.1). This solution satisfies

$$\int_D \int_V f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \leqslant \int_D \int_V F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}, \qquad t \in \mathbb{R}_+ \tag{1.9}$$

If $L(\mathbf{x}, \mathbf{v})\, f(\mathbf{x}, \mathbf{v}, t) \in L^1_+(D \times V)$, then the trace of the solution $f$ satisfies the boundary condition (3) for $t \in \mathbb{R}_+$, a.e. $(\mathbf{x}, \mathbf{v}) \in \Gamma \times V$.

Moreover, *mass conservation*, giving equality in (1.9), i.e., with

$$\int_D \int_V f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} = \int_D \int_V F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}, \qquad t \in \mathbb{R}_+ \tag{1.10}$$

holds together with *uniqueness* (in the relevant $L^1$-space) under some (further) assumptions [for instance, if $Lf \in L^1_+(D \times V)$ and $|\mathbf{nv}|\, f \in L^1(\Gamma \times V)$, or if the detailed balance relations (1.12), (1.17) below hold.[23]

In the rest of this paper we require a *detailed balance* relation (or reciprocity relation) for binary collisions *inside D* between particles with density function $f$ and particles with density function $\psi$, i.e., we assume that there exists a function $E = E(\mathbf{x}, \mathbf{v}) > 0$ such that (see ref. 8, and also ref. 23)

$$K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, E(\mathbf{x}, \mathbf{v}) = K(\mathbf{x}, \mathbf{v}' \to \mathbf{v})\, E(\mathbf{x}, \mathbf{v}') \tag{1.11}$$

or

$$\psi(\mathbf{x}, \mathbf{v}_*)\, E(\mathbf{x}, \mathbf{v}) = \psi(\mathbf{x}, \mathbf{v}'_*)\, E(\mathbf{x}, \mathbf{v}') \tag{1.12}$$

Then by (5) the collision term for $E$ vanishes,

$$(QE)(\mathbf{x}(t), \mathbf{v}(t), t) \equiv 0 \tag{1.13}$$

So, the function $E$ is a "collisionless" solution to Eq. (1) if

$$\frac{d}{dt}(E(\mathbf{x}(t), \mathbf{v}(t))) \equiv 0 \tag{1.14}$$

i.e., if $E$ is constant along the characteristic curves, $E(\mathbf{x}(t), \mathbf{v}(t)) \equiv E(\mathbf{x}(0), \mathbf{v}(0)) \equiv E(\mathbf{y}, \mathbf{u})$, and if $F_0(\mathbf{y}, \mathbf{u}) = E(\mathbf{y}, \mathbf{u})$, $(\mathbf{y}, \mathbf{u}) \in D \times V$. The typical case with detailed balance is given by the local Maxwellian function

$$\mathbf{E}(\mathbf{x}, \mathbf{v}) = \rho_0(\mathbf{x}) \exp(-cm|\mathbf{v}|^2), \qquad \mathbf{x} \in D, \quad \mathbf{v} \in V \tag{1.15}$$

where $\rho_0 \geqslant 0$ is a given function, $c$ is a positive constant, and $m$ is the mass of a particle, if the other (given) density function is

$$\psi(\mathbf{x}, \mathbf{v}_*) = X(\mathbf{x}) \exp(-cm_*\, |\mathbf{v}_*|^2) \tag{1.16}$$

where $X \geqslant 0$ is a given function and $m_*$ is the corresponding particle mass. Here relation (1.12) follows from the energy conservation law for a binary collision. In Section 2 we will prove that there are (essentially) no other functions than the Maxwellians (1.15) and (1.16) which satisfy (1.12).

In the rest of this paper we also assume that the function $E(\mathbf{x}, \mathbf{v})$ satisfies a *detailed balance* relation at the *boundary*,[8,23]

$$|\mathbf{n}\mathbf{v}'|\, R(\mathbf{x}, \mathbf{v}' \to \mathbf{v}, t)\, E(\mathbf{x}, \mathbf{v}') = |\mathbf{n}\mathbf{v}|\, R(\mathbf{x}, -\mathbf{v} \to -\mathbf{v}', t)\, E(\mathbf{x}, -\mathbf{v})$$
$$\mathbf{n}\mathbf{v}' > 0, \qquad \mathbf{n}\mathbf{v} < 0 \tag{1.17}$$

One finds, by straightforward calculations, that $E(\mathbf{x}, \mathbf{v})$ satisfies the boundary condition (3) if (1.17) holds. Then $E(\mathbf{x}(t), \mathbf{v}(t))$ is a solution to the linear Boltzmann equation in the strong form (1) with (2) and (3), and also to the equation in the mild form (1.5) and in the exponential form (1.6). In the special case of an external *potential force* $\mathbf{a} = -\text{grad}_x\, \phi(\mathbf{x})$ we observe that a solution is $E(\mathbf{x}, \mathbf{v}) = \rho_0 \cdot \exp\{-cm[|\mathbf{v}|^2 + 2\phi(\mathbf{x})]\}$, where $m[|\mathbf{v}|^2 + 2\phi(\mathbf{x})]/2$ represents the total energy of a particle.

*Remark*. Many of the results in this paper can be generalized to more general cases with suitable functions $E(\mathbf{x}, \mathbf{v})$ and $\mathbf{a}(\mathbf{x}, \mathbf{v})$. For instance, the coefficient $c$ in (1.15) may depend on $\mathbf{x}$, $c = c(\mathbf{x})$, if $c$, $\phi$, and $\mathbf{a}$ are related, such that

$$\mathbf{v} \cdot \text{grad}_{\mathbf{x}} E + \mathbf{a} \cdot \text{grad}_{\mathbf{v}} E = 0$$

Furthermore, we can also (without any essential changes) study, e.g., external electromagnetic forces $\mathbf{a}(\mathbf{x}, \mathbf{v}) = \mathbf{a}(\mathbf{x}) + \mathbf{v} \times \mathbf{b}(\mathbf{x})$,[10] because $[\mathbf{v} \times \mathbf{b}(\mathbf{x})] \cdot \text{grad}_{\mathbf{v}} E = 0$; cf. (1.15).

We end this section with a lemma which is important in the following sections and concerns the mild solutions $f^q(\mathbf{x}, \mathbf{v}, t)$ given by initial functions $F_0^q$ with a cutoff.

**Lemma 1.1.** Suppose that the detailed balance relations (1.12) and (1.17) hold. Let (for $q = 1, 2, 3,...$)

$$F_0^q(\mathbf{x}, \mathbf{v}) = \min(F_0(\mathbf{x}, \mathbf{v}), q \cdot E(\mathbf{x}, \mathbf{v})), \qquad \mathbf{x} \in D, \quad \mathbf{v} \in V \tag{1.18}$$

Then the mild solution $f^q(\mathbf{x}, \mathbf{v}, t)$ (given by the theorem above) satisfies (for $q = 1, 2, 3,...$)

$$f^q(\mathbf{x}, \mathbf{v}, t) \leqslant q \cdot E(\mathbf{x}, \mathbf{v}), \qquad \mathbf{x} \in D, \quad \mathbf{v} \in V, \quad t \in \mathbb{R}_+ \tag{1.19}$$

*Proof.* Define the iterate functions $f_n^q(\mathbf{x}, \mathbf{v}, t)$ by (1.7). Then by induction we find that

$$f_n^q(\mathbf{x}(t), \mathbf{v}(t), t) \leqslant qE(\mathbf{x}(t), \mathbf{v}(t)), \qquad n \in \mathbb{N}$$

using that $E$ is a (mild) solution. Let $n \to \infty$; then $f_n^q \uparrow f^q$, and (1.19) follows. ∎

## 2. ON THE SUMMATIONAL COLLISION INVARIANTS FOR THE LINEAR BOLTZMANN EQUATION

One of the basic ingredients in kinetic theory is the concept of collision invariants. In the case of the linear Boltzmann equation, the problem deals with finding all functions $E = E(\mathbf{v})$ and $\psi = \psi(\mathbf{v}_*)$, such that the following relation holds:

$$E(\mathbf{v}) \cdot \psi(\mathbf{v}_*) = E(\mathbf{v}') \cdot \psi(\mathbf{v}'_*) \tag{2.1}$$

for all vectors $\mathbf{v}, \mathbf{v}_*, \mathbf{v}', \mathbf{v}'_*$ (representing the velocities in a binary collision) satisfying

$$m\mathbf{v} + m_*\mathbf{v}_* = m\mathbf{v}' + m_*\mathbf{v}'_* \tag{2.2}$$

and

$$m|\mathbf{v}|^2 + m_*|\mathbf{v}_*|^2 = m|\mathbf{v}'|^2 + m_*|\mathbf{v}'_*|^2$$

The problem (2.1) with (2.2) can equivalently be described as that of finding all functions $R = \log E(\mathbf{v})$, $S = \log \psi(\mathbf{v}_*)$, such that

$$R(\mathbf{v}) + S(\mathbf{v}_*) = R(\mathbf{v}') + S(\mathbf{v}'_*) \tag{2.3}$$

holds for all vectors satisfying (2.2).

Already Boltzmann proved[5] that in the case of $C^2$-functions the only solutions to (2.3) are given by

$$\begin{aligned} R(\mathbf{v}) &= -\frac{am}{2} |\mathbf{v}|^2 + \mathbf{b} \cdot (m\mathbf{v}) + C_1 \\ S(\mathbf{v}_*) &= -\frac{am_*}{2} |\mathbf{v}_*|^2 + \mathbf{b} \cdot (m_* \mathbf{v}_*) + C_2 \end{aligned} \tag{2.4}$$

so $E$ and $\psi$ must be Maxwellian functions,

$$\begin{aligned} E(\mathbf{v}) &= E_0 \exp\left[ -\frac{am}{2} |\mathbf{v}|^2 + \mathbf{b} \cdot (m\mathbf{v}) \right] \\ \psi(\mathbf{v}_*) &= \psi_0 \exp\left[ -\frac{am_*}{2} |\mathbf{v}_*|^2 + \mathbf{b} \cdot (m_* \mathbf{v}_*) \right] \end{aligned} \tag{2.5}$$

Here the constants $a, C_1, C_2, E_0, \psi_0 \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^3$ may depend on the space variable $\mathbf{x}$ and the time variable $t$.

We want to solve the problem (2.3), or equivalently (2.1), with (2.2) in the case when the equation holds only almost everywhere in $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. That is a suitable setting for studying the convergence to equilibrium of $L^1$-solutions to the linear Boltzmann equation; see Section 4.

The analogous problem of finding all (summational) collision invariants for the nonlinear Boltzmann equation has recently been studied in the a.e. case by Arkeryd and Cercignani.[4] That is the problem of finding all functions $\phi$ such that the equation

$$\phi(\mathbf{v}) + \phi(\mathbf{v}_*) - \phi(\mathbf{v}') - \phi(\mathbf{v}'_*) = 0 \tag{2.6}$$

holds a.e. (in $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$) for vectors satisfying

$$\mathbf{v} + \mathbf{v}_* = \mathbf{v}' + \mathbf{v}'_* \tag{2.7}$$

and

$$|\mathbf{v}|^2 + |\mathbf{v}_*|^2 = |\mathbf{v}'|^2 + |\mathbf{v}'_*|^2$$

They proved that the general measurable solution of (2.6) is

$$\phi(\mathbf{v}) = A + \mathbf{B} \cdot \mathbf{v} + C|\mathbf{v}|^2 \tag{2.8}$$

with constants $A, C \in \mathbb{R}$, $\mathbf{B} \in \mathbb{R}^3$ (see also ref. 31).

To get this result, they studied an equivalent problem related to (2.6), namely seeking for all functions $\varphi$ such that

$$\varphi(\mathbf{q}_1+\mathbf{q}_2)=\varphi(\mathbf{q}_1)+\varphi(\mathbf{q}_2) \tag{2.9}$$

holds a.e. for $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{R}^3$ with $\mathbf{q}_1 \cdot \mathbf{q}_2 = 0$. Here

$$\varphi(\mathbf{q})=\phi(\boldsymbol{\xi}+\mathbf{q})-\phi(\boldsymbol{\xi}) \tag{2.10}$$

and

$$\begin{aligned} \mathbf{v}' &= \boldsymbol{\xi}+\mathbf{q}_1 \\ \mathbf{v}'_* &= \boldsymbol{\xi}+\mathbf{q}_2 \\ \mathbf{v}_* &= \boldsymbol{\xi}+\mathbf{q}_1+\mathbf{q}_2 \end{aligned} \tag{2.11}$$

We will use the results from ref. 4 to solve Eq. (2.3) for the linear Boltzmann equation with two unknown functions $R$ and $S$. This will be done by some suitable substitutions from problem (2.3) to (2.6) or (2.9).

We start with the following well-known relations for the velocities in a binary collision between particles with masses $m$ and $m_*$,[8,22]

$$\begin{aligned} \mathbf{v}' &= \mathbf{v} - \kappa w \cos\theta \cdot \mathbf{n} \equiv \mathbf{v} - \kappa(\mathbf{n}\mathbf{w}) \cdot \mathbf{n} \\ \mathbf{v}'_* &= \mathbf{v}_* + \kappa_* w \cos\theta \cdot \mathbf{n} \equiv \mathbf{v}_* + \kappa_*(\mathbf{n}\mathbf{w}) \cdot \mathbf{n} \end{aligned} \tag{2.12}$$

where $\kappa = 2m_*/(m+m_*)$, $\kappa_* = 2m/(m+m_*)$, $\mathbf{w} = \mathbf{v} - \mathbf{v}_*$, and $\mathbf{n} = (\mathbf{v}-\mathbf{v}')/|\mathbf{v}-\mathbf{v}'| = (\sin\theta\cos\zeta, \sin\theta\sin\zeta, \cos\theta)$.

One finds that $\mathbf{v}' = \mathbf{v}'(\theta,\zeta)$ and $\mathbf{v}'_* = \mathbf{v}'_*(\theta,\zeta)$ terminate on two concentric spheres with radius $\kappa w/2$ and $\kappa_* w/2$, respectively (see Fig. 1 in ref. 22).

In order to transform our problem (2.3), we first write

$$\mathbf{v}' = \mathbf{v} + \kappa\mathbf{q}_1 \equiv \mathbf{v} + \mathbf{q}_1 + \alpha\mathbf{q}_1$$

where $\mathbf{q}_1 = -(\mathbf{n}\mathbf{w})\mathbf{n}$, and $\alpha = \kappa - 1 = (m_* - m)/(m_* + m)$. Let $\mathbf{q}_2$ be defined, such that

$$\mathbf{v}_* = \mathbf{v} + \mathbf{q}_1 + \mathbf{q}_2$$

Then $\mathbf{q}_2$ is orthogonal to $\mathbf{q}_1$ because

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = \mathbf{q}_1(\mathbf{v}_* - \mathbf{v} - \mathbf{q}_1) = \mathbf{q}_1(-\mathbf{w}) - \mathbf{q}_1^2 = (\mathbf{n}\mathbf{w})^2 - (\mathbf{n}\mathbf{w})^2 = 0$$

We also see that

$$\mathbf{v}'_* = \mathbf{v}_* - \kappa_*\mathbf{q}_1 = \mathbf{v} + \mathbf{q}_2 + \alpha\mathbf{q}_1$$

Summarizing, we find that the velocities in a binary collision are characterized by

$$\begin{aligned} \mathbf{v}' &= \mathbf{v} + \mathbf{q}_1 + \alpha \mathbf{q}_1 \\ \mathbf{v}'_* &= \mathbf{v} + \mathbf{q}_2 + \alpha \mathbf{q}_1 \\ \mathbf{v}_* &= \mathbf{v} + \mathbf{q}_1 + \mathbf{q}_2 \end{aligned} \tag{2.13}$$

with $\alpha = (m_* - m)/(m_* + m)$ and $\mathbf{q}_1 \perp \mathbf{q}_2$ (where $\mathbf{q}_1$ and $\mathbf{q}_2$ are independent of the masses $m$ and $m_*$).

Furthermore, to transform our problem (2.3) to that in ref. 4, let us take a linear transformation $\mathbb{R}^6 \to \mathbb{R}^6$: $(\mathbf{v}, \mathbf{v}_*) \mapsto (\tilde{\mathbf{v}}, \tilde{\mathbf{v}}_*)$ given by

$$\begin{aligned} \tilde{\mathbf{v}} &= \mathbf{v} - \frac{\alpha}{2} \mathbf{w} \equiv \mathbf{v} - \frac{m_* - m}{2(m_* + m)} (\mathbf{v} - \mathbf{v}_*) \\ \tilde{\mathbf{v}}_* &= \mathbf{v}_* - \frac{\alpha}{2} \mathbf{w} \equiv \mathbf{v}_* - \frac{m_* - m}{2(m_* + m)} (\mathbf{v} - \mathbf{v}_*) \end{aligned} \tag{2.14}$$

Let also $\tilde{\mathbf{v}}' = \tilde{\mathbf{v}} + \mathbf{q}_1$ and $\tilde{\mathbf{v}}'_* = \tilde{\mathbf{v}} + \mathbf{q}_2$. Then it follows that this transformed velocities

$$\begin{aligned} \tilde{\mathbf{v}}' &= \tilde{\mathbf{v}} + \mathbf{q}_1 \\ \tilde{\mathbf{v}}'_* &= \tilde{\mathbf{v}} + \mathbf{q}_2 \\ \tilde{\mathbf{v}}_* &= \tilde{\mathbf{v}} + \mathbf{q}_1 + \mathbf{q}_2 \end{aligned} \tag{2.15}$$

satisfy the following (reduced) moment and energy relations [cf. (2.7)]:

$$\tilde{\mathbf{v}} + \tilde{\mathbf{v}}_* = \tilde{\mathbf{v}}' + \tilde{\mathbf{v}}'_*$$

and

$$|\tilde{\mathbf{v}}|^2 + |\tilde{\mathbf{v}}_*|^2 = |\tilde{\mathbf{v}}'|^2 + |\tilde{\mathbf{v}}'_*|^2 \tag{2.16}$$

One also finds that the velocities $\tilde{\mathbf{v}}'$ and $\tilde{\mathbf{v}}'_*$ terminate on a common sphere with radius $w/2$. Then define functions $\tilde{R}$ and $\tilde{S}$ such that

$$\tilde{R}(\tilde{\mathbf{v}}') = R(\mathbf{v}'), \qquad \tilde{S}(\tilde{\mathbf{v}}'_*) = S(\mathbf{v}'_*) \tag{2.17}$$

where

$$\tilde{R}(\tilde{\mathbf{v}}) = R(\mathbf{v}), \qquad \tilde{S}(\tilde{\mathbf{v}}_*) = S(\mathbf{v}_*)$$

Let us now study the problem of finding all measurable functions $\tilde{R} = \tilde{R}(\tilde{\mathbf{v}})$ and $\tilde{S} = \tilde{S}(\tilde{\mathbf{v}}_*)$ such that the following relation holds:

$$\tilde{R}(\tilde{\mathbf{v}}') + \tilde{S}(\tilde{\mathbf{v}}'_*) - \tilde{R}(\tilde{\mathbf{v}}) - \tilde{S}(\tilde{\mathbf{v}}_*) = 0 \tag{2.18}$$

a.e. in $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$, where Eq. (2.15) hold with $\mathbf{q}_1 \cdot \mathbf{q}_2 = 0$. Furthermore, define (for suitable $\xi \in \mathbb{R}^3$) functions

$$\begin{aligned} \tilde{r}(\mathbf{q}) &= \tilde{R}(\xi + \mathbf{q}) - \tilde{R}(\xi) \\ \tilde{s}(\mathbf{q}) &= \tilde{S}(\xi + \mathbf{q}) - \tilde{S}(\xi) \end{aligned} \tag{2.19}$$

Then the problem (2.18) can be written in the following way (for suitable $\xi \in \mathbb{R}^3$): Find all functions $\tilde{r}$ and $\tilde{s}$ such that

$$\tilde{s}(\mathbf{q}_1 + \mathbf{q}_2) = \tilde{r}(\mathbf{q}_1) + \tilde{s}(\mathbf{q}_2) \tag{2.20}$$

holds for almost all $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{R}^3$ with

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = 0$$

Here, with $\mathbf{q} = q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + q_3 \mathbf{e}_3$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ give an orthogonal basis for $\mathbb{R}^3$, it follows by (2.20) that

$$\tilde{s}(\mathbf{q}) = \begin{cases} \tilde{r}(q_1 \mathbf{e}_1) + \tilde{r}(q_2 \mathbf{e}_2) + \tilde{s}(q_3 \mathbf{e}_3) \\ \tilde{r}(q_2 \mathbf{e}_2) + \tilde{r}(q_3 \mathbf{e}_3) + \tilde{s}(q_1 \mathbf{e}_1) \\ \tilde{r}(q_3 \mathbf{e}_3) + \tilde{r}(q_1 \mathbf{e}_1) + \tilde{s}(q_2 \mathbf{e}_2) \end{cases} \tag{2.21}$$

Pairwise subtraction in (2.21) gives that

$$\tilde{r}(q_3 \mathbf{e}_3) - \tilde{s}(q_1 \mathbf{e}_1) = \tilde{r}(q_2 \mathbf{e}_2) - \tilde{s}(q_2 \mathbf{e}_2) = \tilde{r}(q_3 \mathbf{e}_3) - \tilde{s}(q_3 \mathbf{e}_3) = 0$$

Then it follows (for a.e. $\mathbf{q} = q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + q_3 \mathbf{e}_3$) that

$$\tilde{s}(\mathbf{q}) = \tilde{r}(\mathbf{q}) \tag{2.22}$$

So we get, by (2.20) and (2.22), the following equation:

$$\tilde{s}(\mathbf{q}_1 + \mathbf{q}_2) = \tilde{s}(\mathbf{q}_1) + \tilde{s}(\mathbf{q}_2), \qquad \mathbf{q}_1 \cdot \mathbf{q}_2 = 0 \tag{2.23}$$

i.e., the same Cauchy equation as in ref. 4.

Therefore, using the results in ref. 4, we find that the measurable solutions are given by

$$\tilde{r}(\mathbf{q}) = \tilde{s}(\mathbf{q}) = \mathbf{B} \cdot \mathbf{q} + C |\mathbf{q}|^2$$

with constants $\mathbf{B} \in \mathbb{R}^3$, $C \in \mathbb{R}$. Then, by (2.19) it follows that

$$\begin{aligned} \tilde{R}(\mathbf{q}) &= A_1 + \mathbf{B} \cdot \mathbf{q} + C |\mathbf{q}|^2 \\ \tilde{S}(\mathbf{q}) &= A_2 + \mathbf{B} \cdot \mathbf{q} + C |\mathbf{q}|^2 \end{aligned} \tag{2.24}$$

Furthermore, by (2.17) one finds that also the functions $R(\mathbf{v})$ and $S(\mathbf{v}_*)$ are polynomials of second order and then also $C^2$-functions, satisfying the results in ref. 5 [cf. (2.4)].

Summarizing and using (2.14), (2.17), and (2.24), we have proved the following theorem on summation invariants for the linear Boltzmann equation (in the almost-everywhere case).

**Theorem 2.1.** If (2.3) holds for almost everywhere finite, measurable functions in $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$ with vectors $\mathbf{v}, \mathbf{v}_*, \mathbf{v}', \mathbf{v}'_*$ satisfying (2.2), then the functions $R$ and $S$ are given by the formulas (2.4).

**Corollary 2.2.** The most general solutions of the collision invariant problem (2.1) with (2.2) are given by Maxwellian distribution functions (2.5), even if Eq. (2.1) holds only for almost everywhere finite, measurable functions $E$ and $\psi$.

## 3. A GENERAL *H*-THEOREM FOR CONVEX FUNCTIONS (IN THE CUTOFF CASE)

Suppose that $\varphi = \varphi(z)$, $\mathbb{R}_+ \to \mathbb{R}$ is a *convex* $C^1$-function, and let $E = E(\mathbf{x}, \mathbf{v}) > 0$ be a given function. Then a *general* (relative) *H-functional* $H_E^\varphi(f)$ for the solution $f$ can be defined by

$$H_E^\varphi(f)(t) = \int_D \int_V \varphi(f(\mathbf{x}, \mathbf{v}, t)/E(\mathbf{x}, \mathbf{v}) \cdot E(\mathbf{x}, \mathbf{v}))\, d\mathbf{x}\, d\mathbf{v} \tag{3.1}$$

This functional is a generalization of the usual (negative) relative *entropy* functional with

$$\varphi(z) = z \log z \qquad \text{and} \qquad z = f/E \tag{3.2}$$

(see ref. 23 and also refs. 29, 17, and 20).

In proving a general $H$-theorem for the functional (3.1), we will assume that there exist detailed balance relations for the collisions, both inside $D$ [see (1.11), (1.12)] and at the boundary $\Gamma = \partial D$ [see (1.17)]. These assumptions are satisfied for almost all physically interesting cases; see also the results in Section 2.

The following *generalized H-theorem* for solutions to the linear Boltzmann equation with general boundary conditions states that (under the assumptions on detailed balances) the $H$-functional (3.1) is nonincreasing in time with the changes in time bounded (from above) by a nonpositive term coming from the collision term.

**Theorem 3.1.** Let $f = f(\mathbf{x}, \mathbf{v}, t)$ be the mild solution of problem (1)–(4) with (1.1) and (1.2) given in Theorem 1, and let the detailed balance relations (1.11) and (1.17) hold together with (1.14), and

$E \in L^1(D \times V)$. If $H_E^{\varphi}(F_0)$ exists for a given convex $C^1$-function $\varphi$, $\mathbb{R}_+ \to \mathbb{R}$, then the relative $H$-functional $H_E^{\varphi}(f)(t)$ in (3.1) exists for $t > 0$ and is nonincreasing in time. Moreover,

$$H_E^{\varphi}(f)(t) \leqslant H_E^{\varphi}(F_0) + \int_0^t N_E^{\varphi}(f)(\tau)\, d\tau \tag{3.3}$$

where

$$\begin{aligned} N_E^{\varphi}(f)(t) = \frac{1}{2} \int_D \int_V \int_V & K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, E(\mathbf{x}, \mathbf{v}) \\ & \times \left[ \frac{f(\mathbf{x}, \mathbf{v}', t)}{E(\mathbf{x}, \mathbf{v}')} - \frac{f(\mathbf{x}, \mathbf{v}, t)}{E(\mathbf{x}, \mathbf{v})} \right] \\ & \times \left[ \varphi' \left( \frac{f(\mathbf{x}, \mathbf{v}, t)}{E(\mathbf{x}, \mathbf{v})} \right) - \varphi' \left( \frac{f(\mathbf{x}, \mathbf{v}', t)}{E(\mathbf{x}, \mathbf{v}')} \right) \right] d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}' \leqslant 0 \end{aligned} \tag{3.4}$$

*Proof.* The proof follows our earlier proof in ref. 23 for the case with $\varphi(z) = z \log z$, $z = f/E$, so we will here only outline the main steps.

Start with a double cutoff in the initial function

$$F_0^{k,j}(\mathbf{x}, \mathbf{v}) = \frac{1}{j} E(\mathbf{x}, \mathbf{v}) + \min(F_0(\mathbf{x}, \mathbf{v}), kE(\mathbf{x}, \mathbf{v})), \qquad k, j = 1, 2, 3,... \tag{3.5}$$

and construct iterate functions $f_n^{k,j}(\mathbf{x}, \mathbf{v}, t)$, $n = 0, 1, 2,...$, by (1.7). Then (by differentiation along the characteristics) one finds for $t \in [0, T]$, a.e. $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$, that

$$\begin{aligned} &\frac{d}{dt} (f_n^{k,j}(\mathbf{x}(t), \mathbf{v}(t), t)) + L(\mathbf{x}(t), \mathbf{v}(t))\, f_n^{k,j}(\mathbf{x}(t), \mathbf{v}(t), t) \\ &\quad = \int_V K(\mathbf{x}(t), \mathbf{v}' \to \mathbf{v}(t))\, f_{n-1}^{k,j}(\mathbf{x}(t), \mathbf{v}', t)\, d\mathbf{v}' \end{aligned} \tag{3.6}$$

Multiplying by $\varphi'[f_n^{k,j}(\mathbf{x}(t), \mathbf{v}(t), t)/E(\mathbf{x}(t), \mathbf{v}(t))]$ and using (1.2) and (1.4), we get

$$\begin{aligned} &\frac{d}{dt} \left[ \varphi \left( \frac{f_n^{k,j}(\mathbf{x}(t), \mathbf{v}(t), t)}{E(\mathbf{x}(t), \mathbf{v}(t))} \right) \cdot E(\mathbf{x}(t), \mathbf{v}(t)) \right] \\ &\quad = \int_V [K(\mathbf{x}(t), \mathbf{v}' \to \mathbf{v}(t))\, f_{n-1}^{k,j}(\mathbf{x}(t), \mathbf{v}', t) \\ &\qquad - K(\mathbf{x}(t), \mathbf{v}(t) \to \mathbf{v}')\, f_n^{k,j}(\mathbf{x}(t), \mathbf{v}(t), t)] \\ &\qquad \times \varphi' \left( \frac{f_n^{k,j}(\mathbf{x}(t), \mathbf{v}(t), t)}{E(\mathbf{x}(t), \mathbf{v}(t))} \right) d\mathbf{v}' \end{aligned} \tag{3.7}$$

Then integrating $\iiint \cdots d\mathbf{y}\, d\mathbf{u}\, d\tau$ (along the characteristics), using the Green identity and a change of variables $(\mathbf{y}, \mathbf{u}) \mapsto (\mathbf{x}(t), \mathbf{v}(t))$ (see Hypothesis CP2), we get

$$\begin{aligned}
&\int_D \int_V \varphi\left(\frac{f_n^{k,j}(\mathbf{x}, \mathbf{v}, \tau)}{E(\mathbf{x}, \mathbf{v})}\right) E(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \\
&\quad - \int_D \int_V \varphi\left(\frac{F_0^{k,j}(\mathbf{x}, \mathbf{v})}{E(\mathbf{x}, \mathbf{v})}\right) E(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \\
&= \int_0^t \int_D \int_V \int_V K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, E(\mathbf{x}, \mathbf{v}) \left[\frac{f_{n-1}^{k,j}(\mathbf{x}, \mathbf{v}', \tau)}{E(\mathbf{x}, \mathbf{v}')} - \frac{f_n^{k,j}(\mathbf{x}, \mathbf{v}, \tau)}{E(\mathbf{x}, \mathbf{v})}\right] \\
&\quad \times \varphi'\left(\frac{f_n^{k,j}(\mathbf{x}, \mathbf{v}, \tau)}{E(\mathbf{x}, \mathbf{v})}\right) d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'\, d\tau \\
&\quad - \int_0^t \int_\Gamma \int_V \varphi\left(\frac{f_n^{k,j}(\mathbf{x}, \mathbf{v}, \tau)}{E(\mathbf{x}, \mathbf{v})}\right) E(\mathbf{x}, \mathbf{v}) \cdot (\mathbf{n}\mathbf{v})\, d\sigma\, d\mathbf{v}\, d\tau
\end{aligned} \tag{3.8}$$

where $\mathbf{n}\mathbf{v} > 0$ on $\Gamma_+$, $\mathbf{n}\mathbf{v} < 0$ on $\Gamma_-$, and $d\sigma$ is the surface measure.

By induction $(n = 1, 2, 3, \ldots)$ one finds that the following inequalities hold for $k, j \in \mathbb{N}$, $(\mathbf{x}, \mathbf{v}) \in D \times V$, $t \in \mathbb{R}_+$ (see Lemma 1.1):

$$\frac{1}{j} E(\mathbf{x}, \mathbf{v}) \leqslant f_n^{k,j}(\mathbf{x}, \mathbf{v}, t) \leqslant (k+1)\, E(\mathbf{x}, \mathbf{v}) \tag{3.9}$$

Then, letting $n \to \infty$ and using the dominated convergence theorem, we find that (3.8) holds also for $f^{k,j} = \lim_{n \to \infty} f_n^{k,j}$, due to the fact that $\varphi'(z)$ is bounded for $0 < 1/j \leqslant z = f/E \leqslant k + 1 < \infty$.

Now the first term $(I_1)$ on the right-hand side in (3.8) (with $n \to \infty$), i.e., the collision term, can be written after a change of variables $\mathbf{v} \mapsto \mathbf{v}'$, $\mathbf{v}' \mapsto \mathbf{v}$ and using (1.11), (3.4) in the following way:

$$\begin{aligned}
I_1(t) &= \frac{1}{2} \int_0^t \int_D \int_V \int_V K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, E(\mathbf{x}, \mathbf{v}) \left[\frac{f^{k,j}(\mathbf{x}, \mathbf{v}, \tau)}{E(\mathbf{x}, \mathbf{v})} - \frac{f^{k,j}(\mathbf{x}, \mathbf{v}', \tau)}{E(\mathbf{x}, \mathbf{v}')}\right] \\
&\quad \times \left[\varphi'\left(\frac{f^{k,j}(\mathbf{x}, \mathbf{v}', \tau)}{E(\mathbf{x}, \mathbf{v}')}\right) - \varphi'\left(\frac{f^{k,j}(\mathbf{x}, \mathbf{v}, \tau)}{E(\mathbf{x}, \mathbf{v})}\right)\right] d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'\, d\tau \\
&\equiv \int_0^t N_E^\varphi(f^{k,j})(\tau)\, d\tau \leqslant 0
\end{aligned} \tag{3.10}$$

where $(a - b)[\varphi'(b) - \varphi'(a)] \leqslant 0$ if $\varphi$ is convex.

Furthermore, the second term on the right-hand side of (3.8) (with $n \to \infty$), i.e., the boundary term, can be found to be nonpositive by using, e.g., the Jensen inequality for the convex function $\varphi$ (see ref. 8, p. 115, and also ref. 23). Then (3.3) with (3.4) holds for $f = f^{k,j}$, i.e.,

$$\int_D \int_V \varphi(f^{k,j}/E)\, E\, d\mathbf{x}\, d\mathbf{v} \leqslant \int_D \int_V \varphi(F_0^{k,j}/E)\, E\, d\mathbf{x}\, d\mathbf{v} + \int_0^t N_E^{\varphi}(f^{k,j})(\tau)\, d\tau \tag{3.11}$$

Now letting $k, j \to \infty$, and using Fatou's lemma for the nonnegative function,

$$P(f) = \frac{1}{2} KE \cdot \left[\frac{f(\mathbf{v})}{E(\mathbf{v})} - \frac{f(\mathbf{v}')}{E(\mathbf{v}')}\right] \cdot \left[\varphi'\left(\frac{f(\mathbf{v})}{E(\mathbf{v})}\right) - \varphi'\left(\frac{f(\mathbf{v}')}{E(\mathbf{v}')}\right)\right]$$

we get

$$\int_0^t \int_D \int_V \int_V P(f)\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'\, d\tau \leqslant \liminf_{k,j\to\infty} \int_0^t \int_D \int_V \int_V P(f^{k,j})\, d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}'\, d\tau$$

so

$$\limsup_{k,j\to\infty} \int_0^t N_E^{\varphi}(f^{k,j})(\tau)\, d\tau \leqslant \int_0^t N_E^{\varphi}(f)(\tau)\, d\tau$$

where $f = f(\mathbf{x}, \mathbf{v}, t) = \lim_{k,j\to\infty} f^{k,j}(\mathbf{x}, \mathbf{v}, t)$ exists [see Section 1].

Furthermore using the lower semicontinuity property for functionals of convex functions,[23,24,14] one finds that

$$\int_D \int_V \varphi(f/E)\, E\, d\mathbf{x}\, d\mathbf{v} \leqslant \liminf_{k,j\to\infty} \int_D \int_V \varphi(f^{k,j}/E)\, E\, d\mathbf{x}\, d\mathbf{v}$$

By monotone and dominated convergence we also get

$$\lim_{k,j\to\infty} \int_D \int_V \varphi(F_0^{k,j}/E)\, E\, d\mathbf{x}\, d\mathbf{v} \leqslant \int_D \int_V \varphi(F_0/E)\, E\, d\mathbf{x}\, d\mathbf{v}$$

Summarizing, we find that (3.3) with (3.4) holds for $f = \lim_{k,j\to\infty} f^{k,j}$, and the general $H$-theorem is proved. ∎

*Remark.* Results on $H$-functionals with general convex functions have been obtained in other cases (see, e.g., refs. 20 and 17).

## 4. ON WEAK AND STRONG CONVERGENCE TO EQUILIBRIUM IN THE CUTOFF CASE

The question of (weak and strong) convergence (when $t \to \infty$) to a stationary equilibrium solution has been studied (among others) by

Arkeryd,[1,3] Elmroth,[14] Gustafsson,[16] Wennberg,[30] Desvillettes,[11] and di Perna–Lions[25] for the nonlinear Boltzmann equation. We will study the problem for the linear Boltzmann equation (1)–(5) with external potential force

$$\mathbf{a}(\mathbf{x}) = -\operatorname{grad}_{\mathbf{x}} \phi(\mathbf{x}), \qquad \phi \in C^1(D) \tag{4.1}$$

general collision function $B(\theta, w)$, including both soft and hard inverse collision forces, together with [see (2.5)]

$$\begin{aligned} \psi(\mathbf{x}, \mathbf{v}_*) &= X(\mathbf{x})\, G(\mathbf{v}_*), \qquad X(\mathbf{x}) \in L^\infty(D) \\ G(\mathbf{v}_*) &= \exp(-cm_* |\mathbf{v}_*|^2) \end{aligned} \tag{4.2}$$

(with constant $c > 0$ and mass $m_*$). Then [see (1.14)] we find that the local Maxwellian

$$E(\mathbf{x}, \mathbf{v}) = E_0 \exp[-cm|\mathbf{v}|^2 - 2cm\phi(\mathbf{x})] \tag{4.3}$$

is a stationary solution to (1) if (1.17) holds.

The main result of this section is Theorem 4.6, giving strong $L^1$-convergence to a Maxwellian equilibrium solution when $t \to \infty$. The proof is based on a result about weak convergence to equilibrium, Proposition 4.4, together with a lemma about translation continuity, Lemma 4.5. (For discussions on earlier results on asymptotics for the linear equation, see, e.g., ref. 27, Section XI.12, and ref. 12; see also refs. 6, 15, 18, and 19 for further references.)

First we will here prove a uniqueness result for (mild) equilibrium solutions in the cutoff case. We use the $H$-functional (3.1) with $\varphi(z) = z^2$, $z = f/E$, and the following notations [see (4.2) and (4.3)]:

$$H_E^{(2)}(f)(t) = \int_D \int_V \left[\frac{f(\mathbf{x}, \mathbf{v}, t)}{E(\mathbf{x}, \mathbf{v})}\right]^2 E(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \tag{4.4}$$

$$\begin{aligned} P_E(f)(t) &= \int_D \int_V \int_V K(\mathbf{x}, \mathbf{v} \to \mathbf{v}')\, E(\mathbf{x}, \mathbf{v}) \\ &\quad \times \left|\frac{f(\mathbf{x}, \mathbf{v}', t)}{E(\mathbf{x}, \mathbf{v}')} - \frac{f(\mathbf{x}, \mathbf{v}, t)}{E(\mathbf{x}, \mathbf{v})}\right|^2 d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}' \\ &= \int_D \int_V \int_V \int_\Omega B(\theta, w)\, \psi(\mathbf{x}, \mathbf{v}_*)\, E(\mathbf{x}, \mathbf{v}) \\ &\quad \times \left|\frac{f(\mathbf{x}, \mathbf{v}', t)}{E(\mathbf{x}, \mathbf{v}')} - \frac{f(\mathbf{x}, \mathbf{v}, t)}{E(\mathbf{x}, \mathbf{v})}\right|^2 d\mathbf{x}\, d\mathbf{v}\, d\mathbf{v}_*\, d\theta\, d\zeta \end{aligned} \tag{4.5}$$

**Proposition 4.1.** Let $\tilde{f}(\mathbf{x}, \mathbf{v})$ be a (mild) equilibrium solution to the linear Boltzmann equation (with $B\psi E > 0$ almost everywhere), such that the $H$-theorem (3.3) holds for $\varphi(z) = z^2$, together with

$$\int_D \int_V \tilde{f}(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} = \int_D \int_V E(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \tag{4.6}$$

Then $\tilde{f} = E$ a.e. in $D \times V$.

*Proof.* Use the $H$-theorem (3.3) and (3.4) with $\varphi(z) = z^2$, $z = \tilde{f}/E$, and $F_0 = \tilde{f}$ [see (4.4) and (4.5)]:

$$H_E^{(2)}(\tilde{f}) + \int_0^t P_E(\tilde{f})(\tau)\, d\tau \leqslant H_E^{(2)}(\tilde{f})$$

Here $P_E(\tilde{f}) \geqslant 0$, so $P_E(\tilde{f}) \equiv 0$, which implies that (a.e. in $D \times V \times V \times \Omega$)

$$\tilde{f}(\mathbf{x}, \mathbf{v}')\, \psi(\mathbf{x}, \mathbf{v}'_*) = \tilde{f}(\mathbf{x}, \mathbf{v})\, \psi(\mathbf{x}, \mathbf{v}_*)$$

Then, by (4.2) and Theorem 2.1 (on collision invariants) it follows that

$$\tilde{f}(\mathbf{x}, \mathbf{v}) = Y(\mathbf{x}) \exp(-cm|\mathbf{v}|^2)$$

with some function $Y(\mathbf{x})$. Using that $Q(\tilde{f})(\mathbf{x}(t), \mathbf{v}(t)) \equiv 0$, one finds that $\tilde{f}(\mathbf{x}(t), \mathbf{v}(t))$ is constant along a characteristic curve. Then, by (4.1) and (4.2), we find that $Y(\mathbf{x}) = Y_0 \exp[-2cm\phi(\mathbf{x})]$, where the constant $Y_0 = E_0$ because of the mass relation (4.6). The result follows. ▌

*Remark 4.2.* To handle the problems of weak and strong convergence to equilibrium (when $t \to \infty$) for general collision functions $B(\theta, w)$, including both soft and hard inverse collision potentials, we can first make a cutoff in the initial data, $F_0^q = \min(F_0, qE)$, $q = 1, 2, 3,...$ [see (1.18)]. Then (see Lemma 1.1) the mild solution $f^q = f^q(\mathbf{x}, \mathbf{v}, t)$ with initial function $F_0^q$ is bounded by $qE(\mathbf{x}, \mathbf{v})$, so all higher moments of $f^q$ are globally bounded in time

$$\int_D \int_V (1 + v^2)^{\sigma/2} f^q(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} \leqslant q \int_D \int_V (1 + v^2)^{\sigma/2} E(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}, \qquad \sigma > 0 \tag{4.7}$$

Furthermore, using mass conservation and the order relation $f^q \leqslant f^{q+1}$, we observe that a solution $f(\mathbf{x}, \mathbf{v}, t)$ will converge (weakly or strongly) in $L^1$ to the (right) Maxwellian function $E(\mathbf{x}, \mathbf{v})$, when $t \to \infty$, if (e.g.) the solution $f^q(\mathbf{x}, \mathbf{v}, t)$ converges to some Maxwellian $E^q(\mathbf{x}, \mathbf{v}) = C_q \cdot E(\mathbf{x}, \mathbf{v})$ with a constant $C_q > 0$. [$C_q = \|F_0^q\|/\|F_0\|$, using the usual $L^1(D \times V)$-norm.] This statement can be seen in the following way: For given $\varepsilon>0$, choose $q_0=q_0(\varepsilon)$ such that $\|F_0-F_0^q\|<\varepsilon/3$ for $q>q_0$. Then, by mass conservation, $\|E-E^q\|=\|f-f^q\|<\varepsilon/3$ for all $t>0$. So, if (for instance) $\|f^q-E^q\|<\varepsilon/3$ for all $t>T(\varepsilon)$, then $\|f-E\|<\varepsilon$ for $t>T(\varepsilon)$. Consequently, we also notice that (to study convergence to equilibrium) we do not need any entropy assumption [of type $F_0\log(F_0/E)\in L^1(D\times V)$], i.e., we can study the problem with general initial data.

In proving our results in this and next section, we will use a lemma concerning lower semicontinuity of a functional of a convex function (of two variables), appearing in the study of convergence of the collision term in the $H$-theorem [see (3.4), (3.5)] (see also ref. 26 for an analogous case).

**Lemma 4.3.** Let $f_n=f_n^q=f_n^q(\mathbf{x},\mathbf{v},t)\leqslant qE(\mathbf{x},\mathbf{v})$, $n, q=1,2,3,\ldots$, and $B_N=B_N(\theta,w)=\min(B(\theta,w),N)$, $N=1,2,3,\ldots$. If the function $f_n$ converges weakly in $L^1$ to $f=f(\mathbf{x},\mathbf{v},t)$, when $n\to\infty$, then [see (4.5)] (for fixed $q$, $N$)

$$\int_0^t\int_D\int_V\int_V\int_\Omega B_N\psi E\left|\frac{f(\mathbf{x},\mathbf{v},\tau)}{E(\mathbf{x},\mathbf{v})}-\frac{f(\mathbf{x},\mathbf{v}',\tau)}{E(\mathbf{x},\mathbf{v}')}\right|^2 d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}_*\,d\Omega\,d\tau$$

$$\leqslant \liminf_{n\to\infty}\int_0^t\int_D\int_V\int_V\int_\Omega B_N \times\psi E\left|\frac{f_n(\mathbf{x},\mathbf{v},\tau)}{E(\mathbf{x},\mathbf{v})}-\frac{f_n(\mathbf{x},\mathbf{v}',\tau)}{E(\mathbf{x},\mathbf{v}')}\right|^2 d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}_*\,d\Omega\,d\tau \tag{4.8}$$

*Proof.* Use, for the convex function $z(\xi,\eta)=(\xi-\eta)^2$, the following elementary inequality:

$$(\xi-\eta)^2\geqslant(a-b)^2+2(a-b)(\xi-a)+2(b-a)(\eta-b) \tag{4.9}$$

with

$$\xi=f_n(\mathbf{v})/E(\mathbf{v}),\qquad \eta=f_n(\mathbf{v}')/E(\mathbf{v}'),\qquad a=f(\mathbf{v})/E(\mathbf{v}),\qquad b=f(\mathbf{v}')/E(\mathbf{v}')$$

Multiplication of (4.9) by $B_N\psi E\equiv B_N(\theta,w)\,\psi(\mathbf{x},\mathbf{v}_*)\,E(\mathbf{x},\mathbf{v})$ and integration gives the result; for instance, we have

$$\lim_{n\to\infty}\iiiint\!\!\int B_N\psi\left(\frac{f(\mathbf{v})}{E(\mathbf{v})}-\frac{f(\mathbf{v}')}{E(\mathbf{v}')}\right)[f_n(\mathbf{v})-f(\mathbf{v})]\,d\mathbf{x}\,d\mathbf{v}\,d\mathbf{v}_*\,d\Omega\,d\tau=0$$

by using the weak convergence assumption together with the bounded functions $B_N\psi$ and $f/E$. ∎

Now we can formulate a result about *weak* $L^1$-convergence to equilibrium for our mild solution $f$, i.e., giving

$$\int_D \int_V g(\mathbf{x}, \mathbf{v})[f(\mathbf{x}, \mathbf{v}, t) - E(\mathbf{x}, \mathbf{v})]\, d\mathbf{x}\, d\mathbf{v} \to 0 \tag{4.10}$$

when $t \to \infty$, for all test functions $g \in L^\infty(D \times V)$.

**Proposition 4.4.** Let $f(\mathbf{x}, \mathbf{v}, t)$ be the mild solution to the linear Boltzmann equation (1.5) with external potential force $\mathbf{a}$ [see (4.1)], kernel function $\psi$ [see (4.2)], and general collision function $B$ together with boundary conditions (3) and detailed balance (1.17). Then, for every $F_0(\mathbf{x}, \mathbf{v}) \in L^1_+(D \times V)$, the solution $f(\mathbf{x}, \mathbf{v}, t)$ converges in weak $L^1$-sense (4.10), when $t \to \infty$, toward a unique Maxwellian function $E(\mathbf{x}, \mathbf{v})$ [see (4.3)] with

$$\int_D \int_V E(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} = \int_D \int_V F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}$$

*Proof.* First approximate the initial function $F_0$ with $F_0^q$ [see (1.18)]; see Remark 4.2. Then, using the $H$-theorem, Theorem 3.1 with $\varphi(z) = z^2$, $z = f/E$, and $f = f^q \leqslant qE$ (Lemma 1.1), we get

$$H_E^{(2)}(f^q)(t) + \int_0^t P_E(f^q)(\tau)\, d\tau \leqslant H_E^{(2)}(F_0^q) \tag{4.11}$$

where $P_E(f^q) \geqslant 0$ [see (4.4) and (4.5)]. So the integral $\int_0^\infty P_E(f^q)(\tau)\, d\tau$ converges, and there is an increasing sequence $\{t_n\}_1^\infty$ such that

$$\lim_{n \to \infty} P_E(f^q)(t_n) = 0 \tag{4.12}$$

Let

$$f_n(\mathbf{x}, \mathbf{v}) = f^q(\mathbf{x}, \mathbf{v}, t_n) \tag{4.13}$$

Then, by a well-known compactness lemma using [see (4.7)] $\iint (1+v)^\sigma f_n\, d\mathbf{x}\, d\mathbf{v} < C_\sigma$ and $\iint f_n \log(f_n/E)\, d\mathbf{x}\, d\mathbf{v} < C_E$ (see Arkeryd[1] and also refs. 23 and 24) there is a subsequence $\{f_{n_i}\}_{i=1}^\infty$ such that $f_{n_i}$ converges weakly to a function $\tilde{f}(\mathbf{x}, \mathbf{v}) \in L^1_+(D \times V)$ when $i \to \infty$.

Now, using Lemma 4.3 concerning lower semicontinuity of convex functionals, it follows that $P_E(\tilde{f}) = 0$. Then, by Theorem 2.1 and (1.10), we get

$$\tilde{f}(\mathbf{x}, \mathbf{v}) \equiv E^q(\mathbf{x}, \mathbf{v}) = C_q \cdot E(\mathbf{x}, \mathbf{v})$$

with a constant $C_q = \|F_0^q\|/\|F_0\| > 0$, and the Maxwellian function $E$ [see (4.3)].

Finally we prove that the solution $f^q(\mathbf{x}, \mathbf{v}, t)$ converges weakly in $L^1$-sense toward $E^q(\mathbf{x}, \mathbf{v})$ when $t \to \infty$. Here we use a contradiction argument.[1] Together with Remark 4.2, this completes the proof. ∎

In order to take the step from *weak* to *strong* $L^1$-convergence (when $t \to \infty$), we will now prove a result, Lemma 4.5, concerning *translation continuity* of our (mild) solution. Then Theorem 4.6, on strong convergence to equilibrium, will follow in the same way as earlier results by Carleman[7] and Gustafsson[16] for the nonlinear space-homogeneous Boltzmann equation.

**Lemma 4.5.** Let $f(\mathbf{x}, \mathbf{v}, t)$ and $f_{\mathbf{h},\mathbf{u}}(\mathbf{x}, \mathbf{v}, t)$ be the mild solutions of the linear Boltzmann equation (1)–(5) with initial data $F_0(\mathbf{x}, \mathbf{v})$ and $F_0(\mathbf{x}+\mathbf{h}, \mathbf{v}+\mathbf{u})$, respectively. Then

$$\lim_{(h+u)\to 0} \int_D \int_V |f_{\mathbf{h},\mathbf{u}}(\mathbf{x}, \mathbf{v}, t) - f(\mathbf{x}, \mathbf{v}, t)| \, d\mathbf{x} \, d\mathbf{v} = 0 \tag{4.14}$$

uniformly in time, $t \in \mathbb{R}_+$.

*Proof.* Let $\varepsilon > 0$ be given. We will approximate the functions in two steps. First approximate the initial function $F_0$ by a continuous function $F_{0,c}^q$ with compact support and bounded by $qE(\mathbf{x}, \mathbf{v})$, such that [with the usual $L^1(D \times V)$-norm]

$$\|F_{0,c}^q - F_0\| < \varepsilon/3, \qquad q > q_0, \quad \text{some } q_0 = q_0(\varepsilon) \tag{4.15}$$

This can be done in the following way: Let first $\tilde{F}_0^q = \min(F_0, qE)$ for $|\mathbf{v}| \leqslant q$ and $\tilde{F}_0^q = 0$ for $|\mathbf{v}| > q$, $q = 1, 2, 3, \ldots$. Now $\tilde{F}_0^q \nearrow F_0$, when $q \to \infty$, and (e.g.) $\|\tilde{F}_0^q - F_0\| < \varepsilon/9$ for $q > q_0$, some $q_0 = q_0(\varepsilon)$. Then use convolution to get a new continuous (approximative) function $\tilde{F}_{0,c}^q$, such that $\|\tilde{F}_{0,c}^q - \tilde{F}_0^q\| < \varepsilon/9$. So, with $F_{0,c}^q = \min(\tilde{F}_{0,c}^q, qE)$, and $q$ large enough, the statement (4.15) follows.

Next, extend the function $F_{0,c}^q$ to a continuous function, defined also in a neighborhood of $D \times V_q$, where $V_q = \{\mathbf{v}: |\mathbf{v}| \leqslant q\}$, such that $F_{0,c}^q$ vanishes outside this extended domain $(\tilde{D} \times \tilde{V}_q)$. Then, using that this continuous function with compact support is uniformly continuous in $\tilde{D} \times \tilde{V}_q$, we can find (for given $\varepsilon > 0$) a $\delta = \delta(\varepsilon) > 0$ such that for all $\mathbf{h}, \mathbf{u}$ with $h^2 + u^2 < \delta^2$ and $\mathbf{x}, \mathbf{v} \in \tilde{D} \times \tilde{V}_q$,

$$|F_{0,c}^q(\mathbf{x}+\mathbf{h}, \mathbf{v}+\mathbf{u}) - F_{0,c}^q(\mathbf{x}, \mathbf{v})| < (\varepsilon/3)\tilde{C}_q \tag{4.16}$$

with a constant $\tilde{C}_q = \min(E(\mathbf{x}, \mathbf{v})/\|E\|) > 0$. Now we get by (4.16) that (for $\mathbf{x}, \mathbf{v} \in \tilde{D} \times \tilde{V}_q$)

$$|F^q_{0,c}(\mathbf{x}+\mathbf{h}, \mathbf{v}+\mathbf{u}) - F^q_{0,c}(\mathbf{x}, \mathbf{v})| < (\varepsilon/3)\, E(\mathbf{x}, \mathbf{v})/\|E\| \tag{4.17}$$

Then, using the linearity of the Boltzmann equation together with mass conservation, we find from (4.17) that (for $\mathbf{x}, \mathbf{v} \in \tilde{D} \times \tilde{V}_q$, and all $t > 0$) there holds

$$|f^c_{\mathbf{h},\mathbf{u}}(\mathbf{x}, \mathbf{v}, t) - f^c(\mathbf{x}, \mathbf{v}, t)| < (\varepsilon/3)\, E(\mathbf{x}, \mathbf{v})/\|E\| \tag{4.18}$$

where $f^c_{\mathbf{h},\mathbf{u}}$ and $f^c$ are the corresponding mild solutions (with initial data $F^q_{0,c}$ and $F^q_{0,c}$). By (4.18) it follows that $\|f^c_{\mathbf{h},\mathbf{u}} - f^c\| < \varepsilon/3$ holds, uniformly in time $t > 0$. Summarizing and using (4.15), we find that $\|f_{\mathbf{h},\mathbf{u}} - f\| < \varepsilon$ holds for all $t > 0$, and all $\mathbf{h}, \mathbf{u}$ with $h^2 + u^2 < \delta^2$, where $f_{\mathbf{h},\mathbf{u}}$ and $f$ are the solutions with initial functions $F_0(\mathbf{x}+\mathbf{h}, \mathbf{v}+\mathbf{u})$ and $F_0(\mathbf{x}, \mathbf{v})$, respectively. So the translation continuity property (4.14) follows. ∎

*Remark.* We observe that the limit (4.14) is also uniform in the cutoffs. This is so, because the estimate $\|f_{\mathbf{h},\mathbf{u}} - f\| < \varepsilon$ holds independently of the cutoff radius; see also Section 5.

Finally we come to the main result in this section concerning strong convergence to equilibrium in the cutoff case.

**Theorem 4.6.** Let $f = f(\mathbf{x}, \mathbf{v}, t)$ be the mild solution to the linear Boltzmann equation in the case of external potential force (4.1), general collision function $B$ (including both soft and hard inverse potentials), and (Maxwellian) kernel function $\psi = \psi(\mathbf{v}_x)$ [see (4.2)], together with general boundary conditions (3) and detailed balance relation (1.17). Then, for every $F_0(\mathbf{x}, \mathbf{v}) \in L^1_+(D \times V)$, the solution $f(\mathbf{x}, \mathbf{v}, t)$ converges strongly in $L^1$, when $t \to \infty$, toward a unique Maxwellian function $E(\mathbf{x}, \mathbf{v})$, see (4.3) (where $\|E\| = \|F_0\|$), i.e.,

$$\lim_{t \to \infty} \int_D \int_V |f(\mathbf{x}, \mathbf{v}, t) - E(\mathbf{x}, \mathbf{v})|\, d\mathbf{x}\, d\mathbf{v} = 0$$

*Proof.* Use the weak convergence result, Proposition 4.4, together with the translation continuity property, Lemma 4.5. Then the theorem follows; see refs. 7 and 16 and also ref. 13. ∎

## 5. THE CASE OF INFINITE-RANGE FORCES WITHOUT CUTOFF

In this section the linear Boltzmann equation is considered without cutoff in the collision term, i.e., including infinite-range forces. It is studied in the following weak form, which can formally be derived from Eq. (1) with (2) and (5):[23]

$$\int_D \int_V g(\mathbf{x}, \mathbf{v}, t) f(\mathbf{x}, \mathbf{v}, t) \, d\mathbf{x} \, d\mathbf{v}$$

$$= \int_D \int_V g(\mathbf{x}, \mathbf{v}, 0) F_0(\mathbf{x}, \mathbf{v}) \, d\mathbf{x} \, d\mathbf{v}$$

$$+ \int_0^t \int_D \int_V \left[ \mathbf{v} \cdot \operatorname{grad}_{\mathbf{x}} g(\mathbf{x}, \mathbf{v}, \tau) + \mathbf{a}(\mathbf{x}, \mathbf{v}) \operatorname{grad}_{\mathbf{v}} g(\mathbf{x}, \mathbf{v}, \tau) + \frac{\partial}{\partial \tau} g(\mathbf{x}, \mathbf{v}, \tau) \right]$$

$$\times f(\mathbf{x}, \mathbf{v}, \tau) \, d\mathbf{x} \, d\mathbf{v} \, d\tau$$

$$+ \int_0^t \int_D \int_V \int_V \int_\Omega [g(\mathbf{x}, \mathbf{v}', \tau) - g(\mathbf{x}, \mathbf{v}, \tau)]$$

$$\times B(\theta, w) \psi(\mathbf{x}, \mathbf{v}_*) f(\mathbf{x}, \mathbf{v}, \tau) \, d\mathbf{x} \, d\mathbf{v} \, d\mathbf{v}_* \, d\theta \, d\zeta \, d\tau \tag{5.1}$$

for all test functions $g \in C_0^{1,\infty}$ (for simplicity). Here $C_0^{1,\infty} = \{ g \in C^{1,\infty}: g(\mathbf{x}, \mathbf{v}, t) = 0, \ \mathbf{x} \in \Gamma = \partial D \}$, where

$$C^{1,\infty} = \Big\{ g \in C^1(D \times V \times [0, \infty)): \| g \|_1 = \sup | g(\mathbf{x}, \mathbf{v}, t) |$$

$$+ \sup \left| \frac{\partial}{\partial t} g(\mathbf{x}, \mathbf{v}, t) \right| + \sup | \operatorname{grad}_{\mathbf{x}} g(\mathbf{x}, \mathbf{v}, t) |$$

$$+ \sup | \operatorname{grad}_{\mathbf{v}} g(\mathbf{x}, \mathbf{v}, t) | < \infty \Big\} \tag{5.2}$$

The mathematical problems in the noncutoff case come from the nonintegrability of the function $B(\theta, w)$ when $\theta \to \pi/2$ (see the Introduction). Here we study (for simplicity) inverse $k$th power potentials with collision function (see the Introduction)

$$B(\theta, w) = w^\gamma b(\theta), \qquad w = |\mathbf{v} - \mathbf{v}_*|, \qquad \gamma = (k-5)/(k-1), \qquad 3 < k < \infty \tag{5.3}$$

where $\int_0^{\pi/2} b(\theta) \, d\theta = \infty$, $\int_0^{\pi/2} b(\theta) \cos \theta \, d\theta < \infty$.

Then we have the following result on the existence of $L^1$-solutions:[23]

**Theorem E.** Let the assumptions on $\psi(\mathbf{x}, \mathbf{v}_*)$ [see (4.2)] and $B(\theta, w)$ [see (5.3)] together with (1.17) be satisfied. Suppose $F_0 \log(F_0/E) \in L^1(D \times V)$. Then there exists (for $t > 0$) a solution

$f(\mathbf{x}, \mathbf{v}, t) \in L^1_+(D \times V)$ to the linear Boltzmann equation in the integral form (5.1). The solution conserves mass,

$$\int_D \int_V f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}\, d\mathbf{v} = \int_D \int_V F_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v} \tag{5.4}$$

Higher moments are globally bounded in time for hard potentials ($k \geqslant 5$) and locally bounded for soft potentials ($3 < k < 5$).

*Remark.* The results in Theorem E can (in some cases) be extended to very soft potentials; see ref. 9 for $2 < k \leqslant 3$, and ref. 10 for $9/5 < k \leqslant 2$.

For our main result in this section, Theorem 5.3, on (weak and) strong convergence to equilibrium for our solutions, an $H$-theorem in the infinite-range case will be used [see (3.1)].

**Proposition 5.1.** Let $f(\mathbf{x}, \mathbf{v}, t)$ be the $L^1$-solution to Eq. (5.1), given by Theorem E and constructed as a limit of mild (cutoff) solutions.

A. If $\varphi: \mathbb{R}_+ \to \mathbb{R}$ is a convex $C^1$-function, and $H^\varphi_E(F_0)$ exists [see (3.1)], then

$$H^\varphi_E(f)(t) \leqslant H^\varphi_E(F_0), \qquad t > 0 \tag{5.5}$$

B. In particular, if $\varphi(z) = z^2$, $z = f/E$, with $f = f^q(\mathbf{x}, \mathbf{v}, t)$ the solution belonging to $F^q_0$ [see (1.18)], then [see (4.4) and (4.5)]

$$H^{(2)}_E(f^q)(t) + \int_0^t P_E(f^q)(\tau)\, d\tau \leqslant H^{(2)}_E(F^q_0) \tag{5.6}$$

*Proof.* Suppose $f_n(\mathbf{x}, \mathbf{v}, t)$ is the mild solution with cutoff radius $r_n = n$, $n = 1, 2, 3,\ldots$. Then, by a compactness lemma,[1,23] there is a subsequence $\{f_{n_j}\}$ converging weakly to a solution $f(\mathbf{x}, \mathbf{v}, t)$ of the infinite-range equation (5.1). From here statement A follows by the lower semicontinuity property for convex functionals (of one variable), together with the $H$-theorem in the cutoff case, Theorem 3.1.

For statement B use for the collision term Lemma 4.3 concerning a lower semicontinuity property for convex functionals (of two variables). Then [using bounded collision functions $B_N = \min(B, N)$, and letting $N \to \infty$; see also ref. 26], we get

$$\iiiint\!\!\int B\psi E \left| \frac{f(\mathbf{v}')}{E(\mathbf{v}')} - \frac{f(\mathbf{v})}{E(\mathbf{v})} \right|^2$$

$$\leqslant \lim_{N \to \infty} \liminf_{j \to \infty} \iiiint\!\!\int B_N \psi E \left| \frac{f_{nj}(\mathbf{v}')}{E(\mathbf{v}')} - \frac{f_{nj}(\mathbf{v})}{E(\mathbf{v})} \right|^2$$

The proposition follows. ∎

*Remark 5.2.* By the proof of Proposition 5.1 the functions $f_{nj}(\mathbf{x}, \mathbf{v}, t)$ converge weakly in $L^1$ to $f(\mathbf{x}, \mathbf{v}, t)$, when $j \to \infty$. Now, using that the translation continuity holds independently of the cutoffs (Lemma 4.5), we find that the convergence (in fact) is strong in $L^1$-sense,

$$\int_D \int_V |f_{nj}(\mathbf{x}, \mathbf{v}, t) - f(\mathbf{x}, \mathbf{v}, t)| \, d\mathbf{x} \, d\mathbf{v} \to 0, \qquad j \to \infty$$

We can now formulate the main result of this section, concerning strong convergence to equilibrium of our solutions in the infinite-range case, for both soft and hard collision potentials, $3 < k < \infty$.

**Theorem 5.3.** Let $f(\mathbf{x}, \mathbf{v}, t)$ be the solution (given by Theorem E) to the weak form of the linear Boltzmann equation (5.1) in the infinite-range case, with $\psi = \psi(\mathbf{v}_*)$ and $B$ given by (4.2), (5.3), together with external force $\mathbf{a}$ [see (4.1)] and general boundary function [see (3), (4), and (1.17)]. Then for every $F_0(\mathbf{x}, \mathbf{v}) \in L^1_+(D \times V)$ the solution $f(\mathbf{x}, \mathbf{v}, t)$ converges in strong $L^1$-sense (when $t \to \infty$) to a unique Maxwellian function $E(\mathbf{x}, \mathbf{v})$ [see (4.3)] with

$$\int_D \int_V E(\mathbf{x}, \mathbf{v}) \, d\mathbf{x} \, d\mathbf{v} = \int_D \int_V F_0(\mathbf{x}, \mathbf{v}) \, d\mathbf{x} \, d\mathbf{v}$$

*Proof.* First approximate $F_0$ in $L^1$ with $F_0^q = \min(F_0, qE)$, $q = 1, 2, 3,...$ (see Remark 4.2). Next use the $H$-theorem (Proposition 5.1B) with $\varphi(z) = z^2$ to prove uniqueness of the stationary solution $E^q(\mathbf{x}, \mathbf{v})$ (Proposition 4.1). Then, continuing as in the cutoff case, the weak convergence result follows, using (among others) the $H$-theorem, Proposition 5.1B. Finally, we get strong $L^1$-convergence to the Maxwellian equilibrium solution when $t \to \infty$ (see the proof of Theorem 4.6). For this we use the weak convergence result, together with the translation continuity property, Lemma 4.5, where the estimates $\|f^{n_j}_{\mathbf{h},\mathbf{u}} - f^{n_j}\| < \varepsilon$ hold independently of the cutoff radius $n_j$ (see Remark 5.2 and the Remark after Lemma 4.5). ∎

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